# Submodular Max-Min Allocation under Identical Valuations

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# Computational Problem

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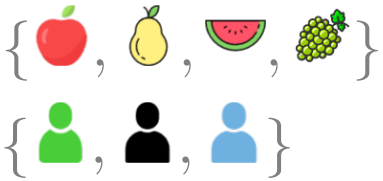

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- A set $J$ of items — $\{🍎, 🍐, 🍉, 🍇\}$
- A set $P$ of players — $\{👤, 👤, 👤\}$
- A valuation function $f : 2^J \to \mathbb{R}$ — $f(\{🍎, 🍉\}) = 5$

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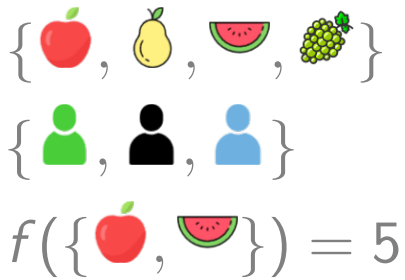

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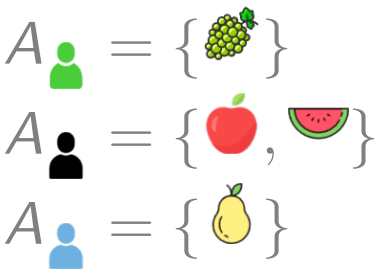

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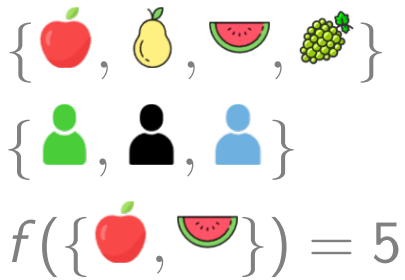


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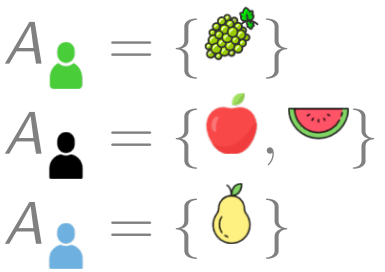

- Classes of valuation functions that are both expressive and computationally tractable?

# Additive and Submodular Functions

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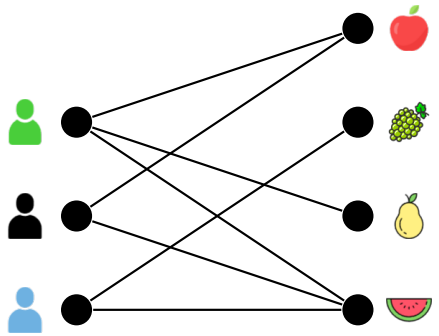
- Restricted assignment: Allocations are restricted by a bipartite graph, $4+\varepsilon$-approximation [Bansal & Sviridenko '06, Asadpour, Kalaitzis, Svensson '08, Cheng, Mao '19]



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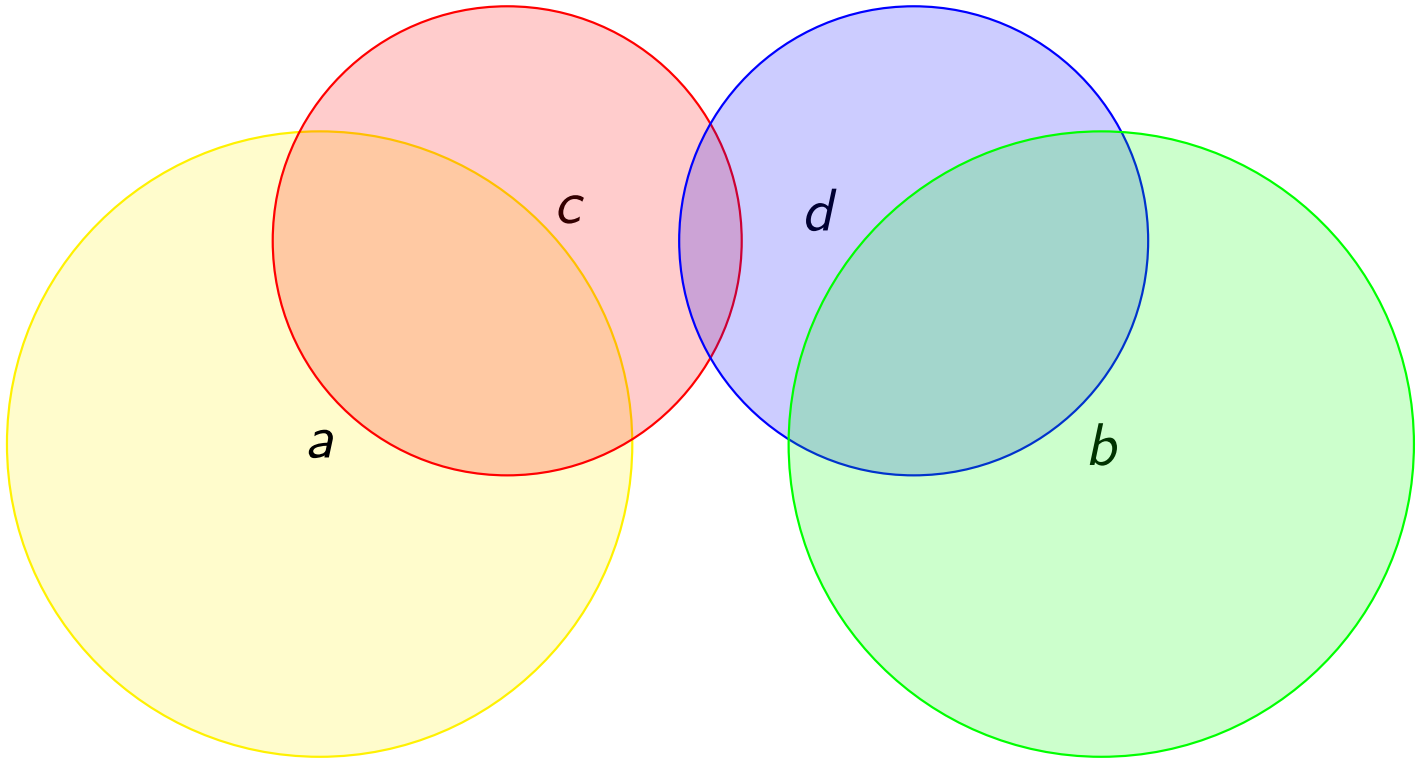

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- Submodular Maximin share: $\approx 0.37$ [Uziahu and Feige '23]

# A Greedy Algorithm

1. Guess the optimal value OPT
2. Allocate the $m$ largest items, one to each player
3. $P' = \{p \in P \mid f(A_p) < \alpha \cdot \mathrm{OPT}\}$
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## Theorem

*Setting $\alpha = \frac{2}{5} = 0.4$, this greedy achieves a $\frac{2}{5}$-approximation. Furthermore, no value of $\alpha$ can achieve an approximation better than 0.4069.*

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- Let $q$ be the player who receives the least in greedy.

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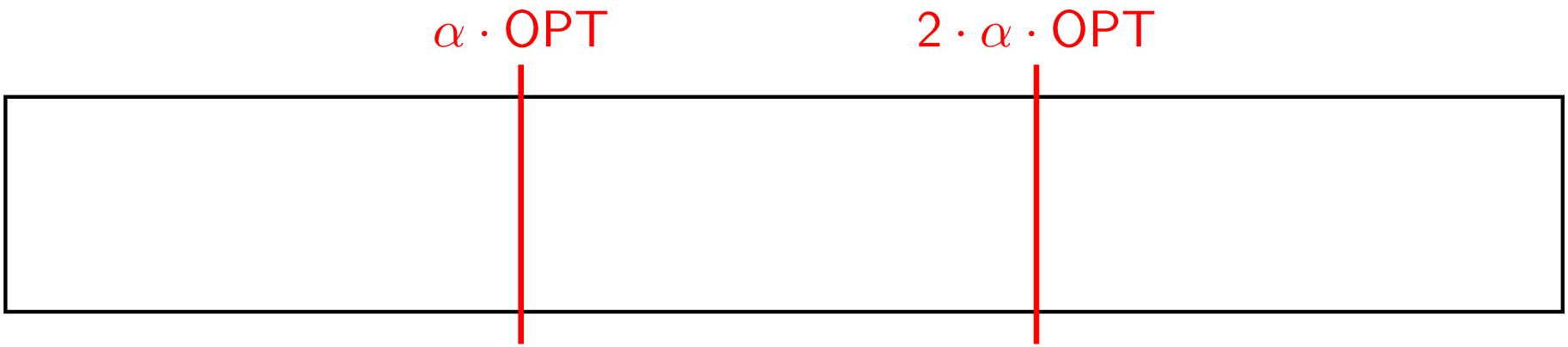

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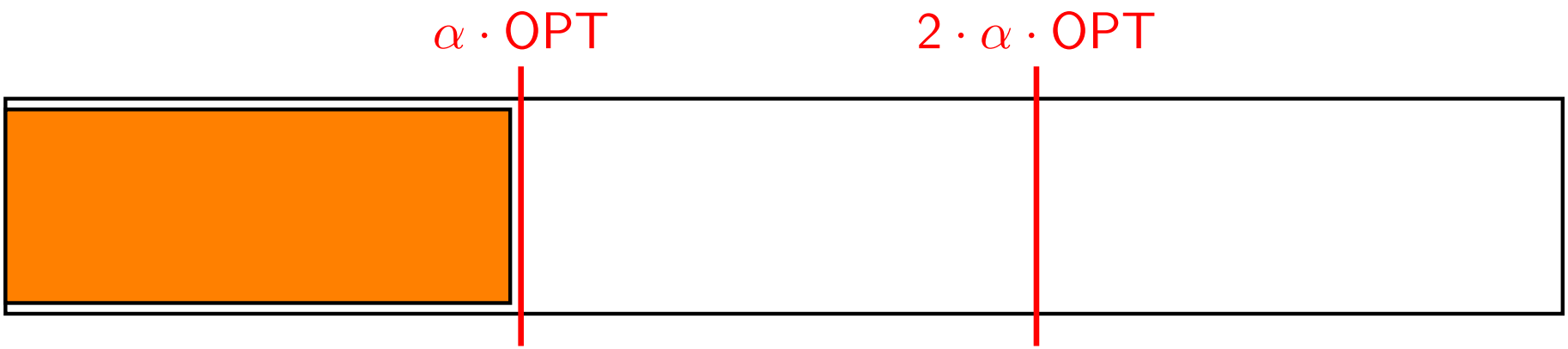

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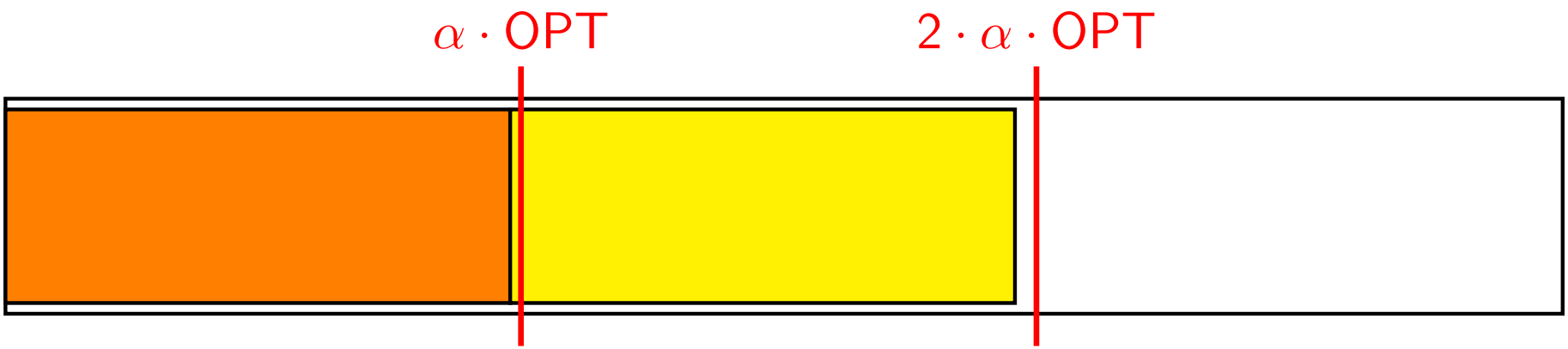

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**Theorem**

*The integrality gap of the configuration LP under identical submodular valuations is between* $\frac{4}{3}$ *and* 3.

# Upper-bounding the integrality gap

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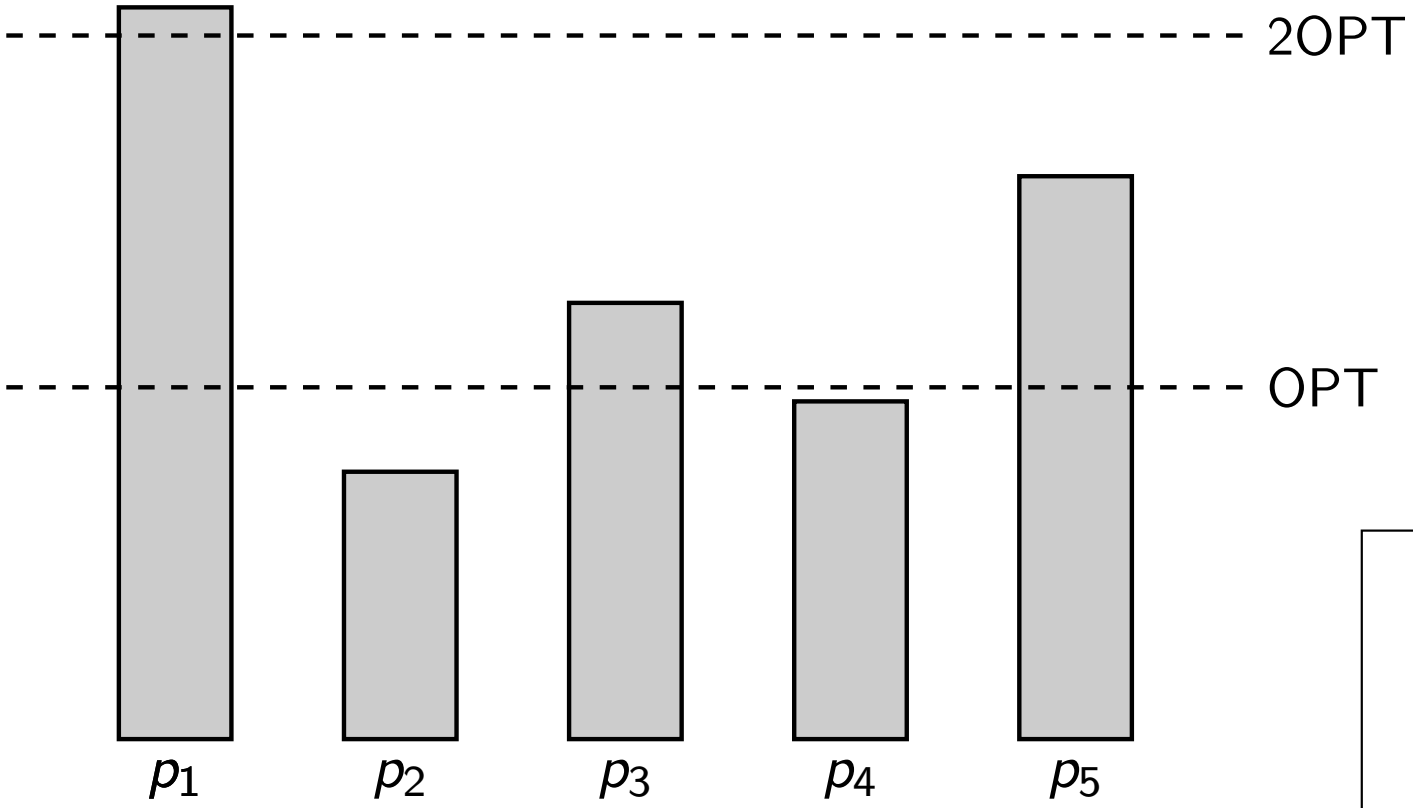
$\Rightarrow$ The integrality gap of CLP is $\leq \beta$.

# Proof sketch

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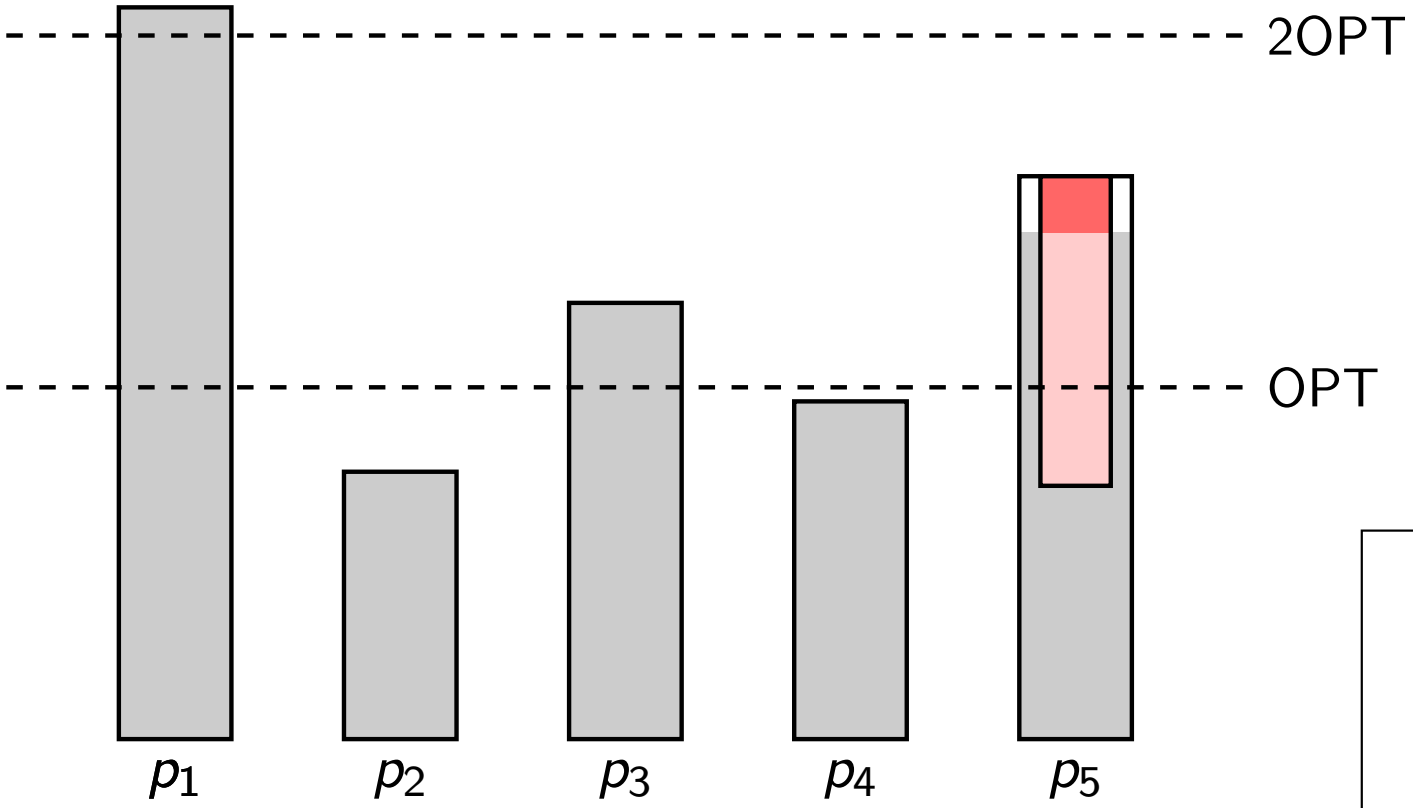


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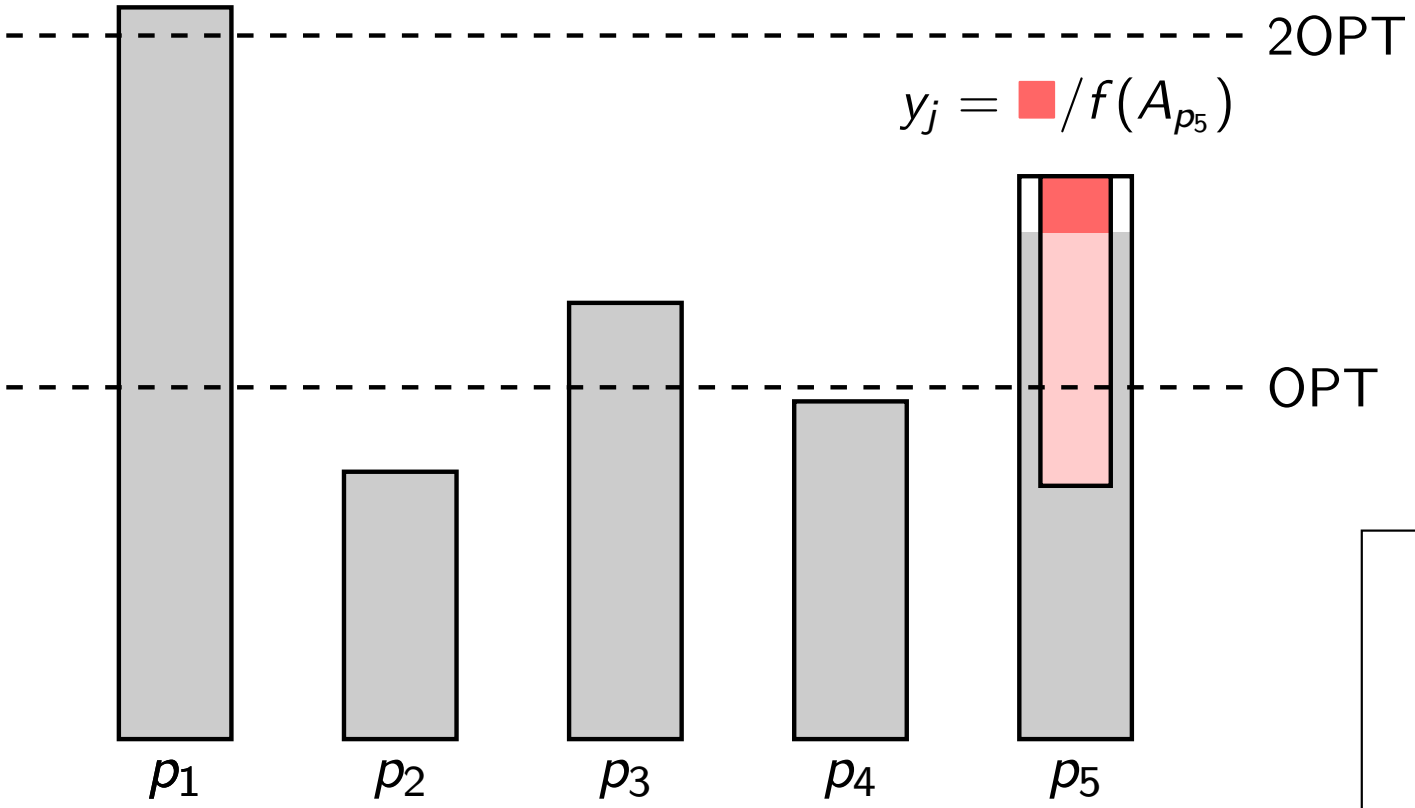


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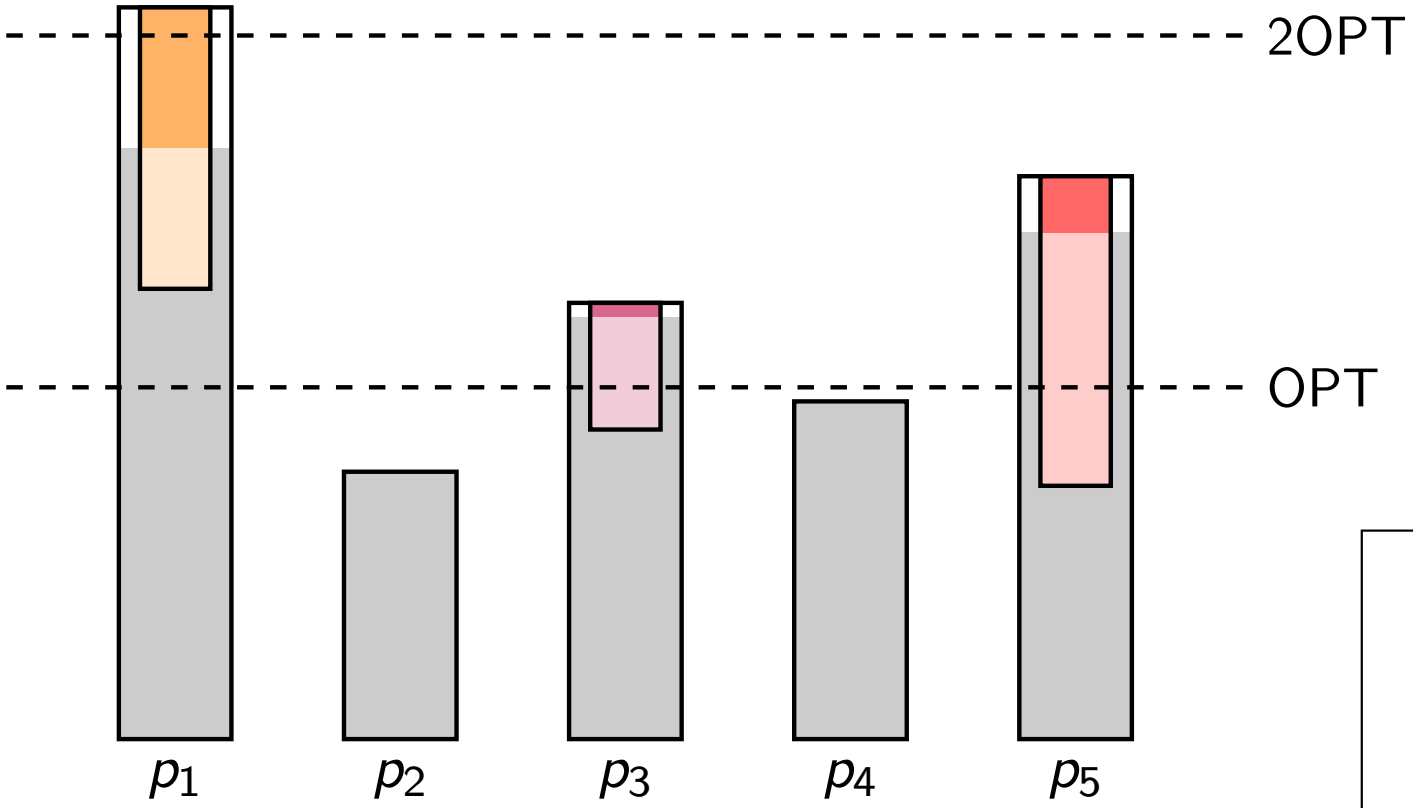


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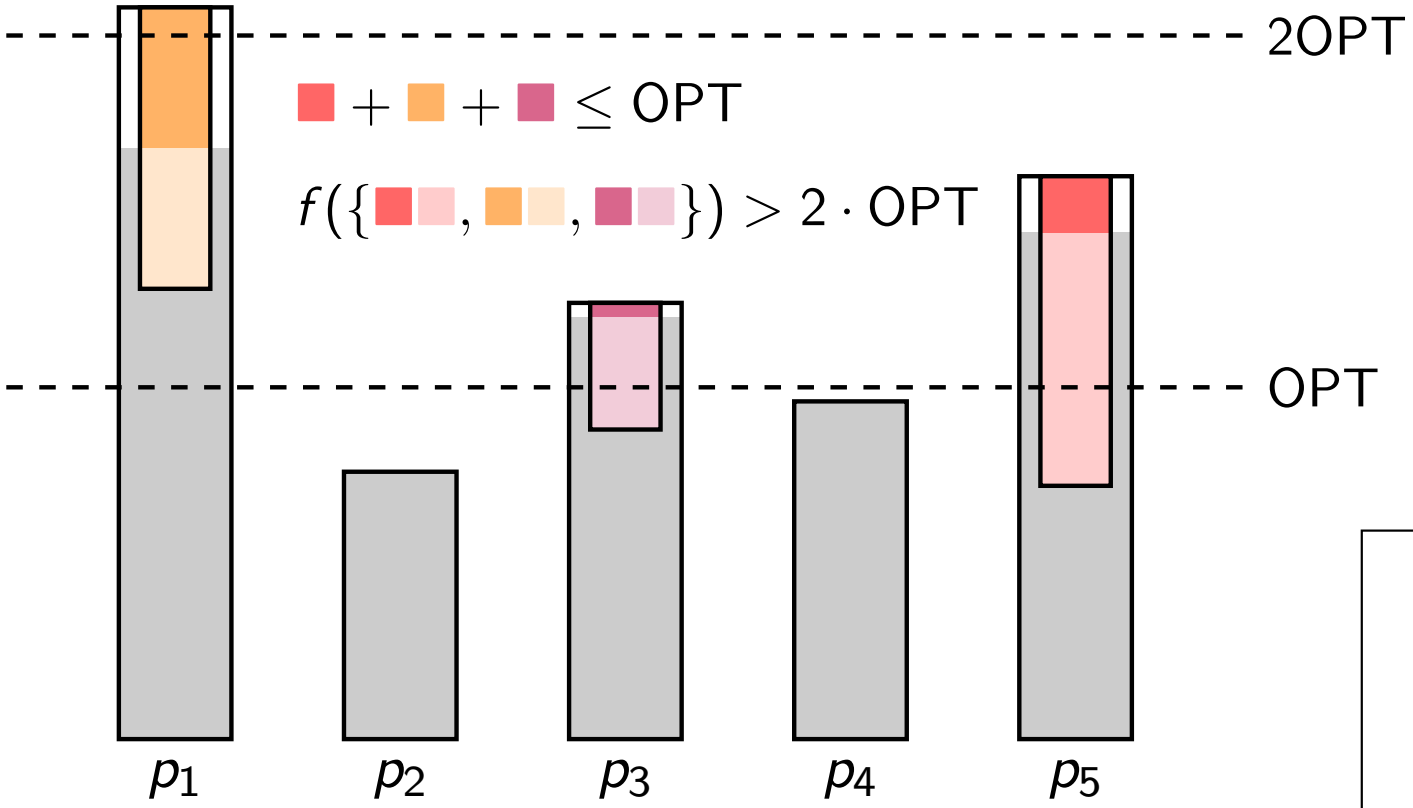


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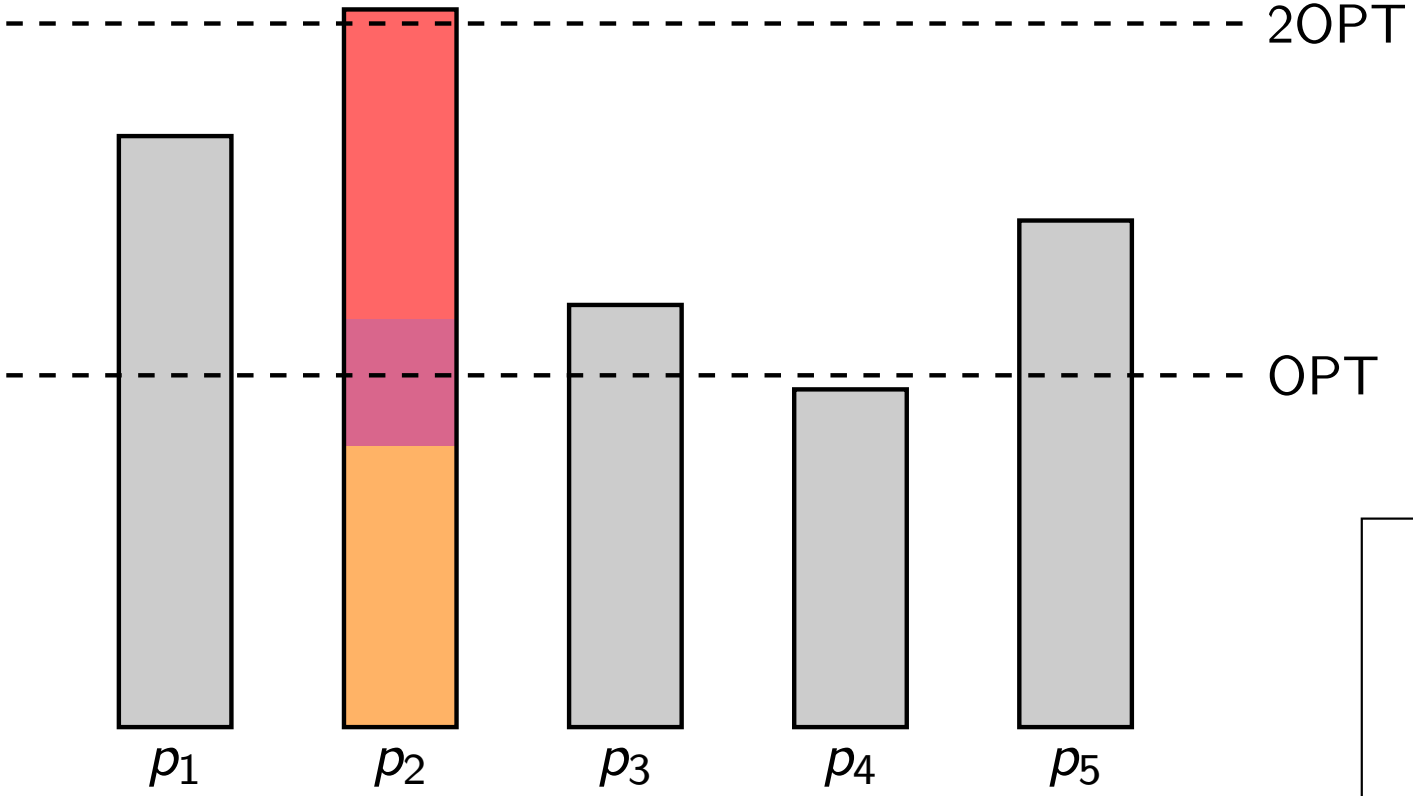


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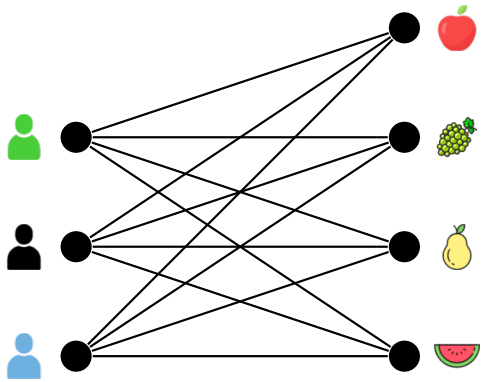

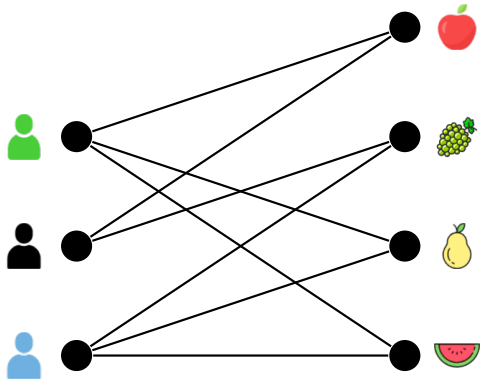

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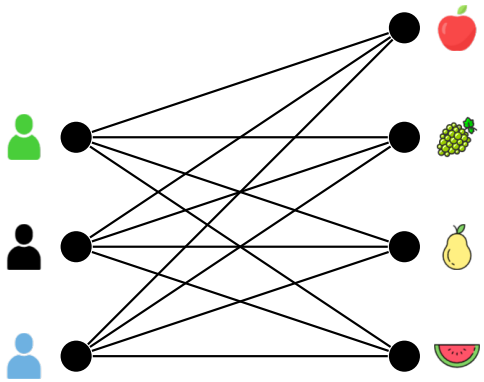
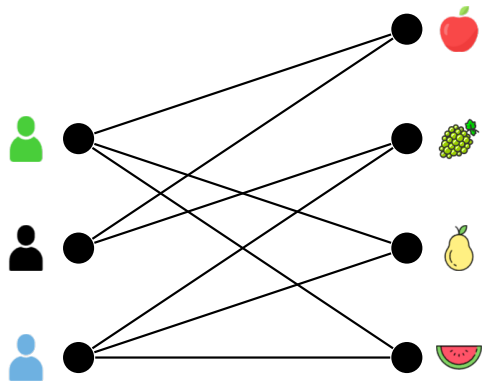


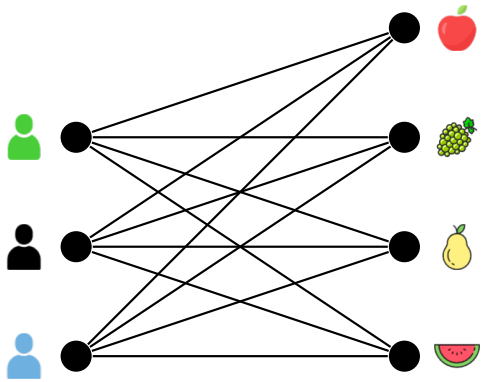


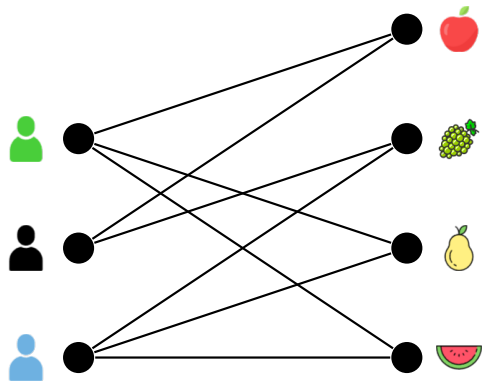
[Chakrabarty, Chuzhoy, Khanna '09]

- Make the bound on the integrality gap constructive

# Conclusion

- Greedy gives a 2/5-approximation. Can we improve this?
- **NP**-hard to approximate better than $1 - \frac{1}{e} \approx 0.63$ [Khot, Lipton, Markakis, Mehta '08]
- Constant integrality gap for CLP with identical valuations. What about more general valuations?





[Chakrabarty, Chuzhoy, Khanna '09]

- Make the bound on the integrality gap constructive

**Thank you for your attention!**